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Sharp Hardy constants for annuli

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ABSTRACT

We consider the Hardy inequality in canonical doubly connected plane domains. For any annulus A we determine sharp Hardy's constant $c_2(A)$ in function of conformal modulus $M(A)$. Namely, for any annulus A with fixed conformal modulus $M(A) = M$ we prove that

$$c_2(A) = \begin{cases} 1/4, & \text{if } M \in (0, M^*]; \\ \gamma(2-\gamma)/4, & \text{if } M \in (M^*, \infty), \end{cases}$$

where $\gamma = \gamma(M) \in (1, 2)$. The critical modulus $M^* \approx 0.57298$ and the values of $\gamma(M)$ are found as roots of certain equations, containing the Gauss hypergeometric functions. In particular, we show that the sharp Hardy constants $c_2(A)$ depend on M continuously and that they tend to zero as $M \to \infty$. In addition, we describe an application of results to a Rellich type inequality.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a plane domain such that $\Omega \neq \mathbb{C}$. We consider functions $\varphi : \Omega \to \mathbb{R}$ and the following Hardy inequality

$$\iint\limits_{\Omega} |\nabla\varphi(z)|^2 dxdy \geq c_2(\Omega) \iint\limits_{\Omega} \frac{\varphi^2(z)}{(\text{dist}(z, \partial\Omega))^2} dxdy, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{1}$$

where $z = x + iy$, $\text{dist}(z, \partial\Omega) := \inf_{\zeta \in \partial\Omega} |z - \zeta|$ is the distance from a point $z \in \Omega$ to the boundary of the domain. We suppose that the quantity $c_2(\Omega)$ is defined as the best possible constant, i.e.

$$c_2(\Omega) = \inf_{\varphi \in C_0^\infty(\Omega), \varphi \neq 0} \frac{\iint_\Omega |\nabla\varphi(z)|^2 dxdy}{\iint_\Omega \varphi^2(z)(\text{dist}(z, \partial\Omega))^{-2} dxdy}. \tag{2}$$

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